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A two-parameter generalization of Shannon-Khinchin Axioms and the uniqueness theorem

Tatsuaki Wada[1,*] and Hiroki Suyari[2,†]

[1] *Department of Electrical and Electronic Engineering, Ibaraki University, Hitachi, Ibaraki 316-8511, Japan*

[2] *Department of Information and Image Sciences, Faculty of Engineering, Chiba University, 263-8522, Japan*

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Based on the one-parameter generalization of Shannon-Khinchin (SK) axioms presented by one of the authors, and utilizing a tree-graphical representation, we have further developed the SK Axioms in accordance with the two-parameter entropy introduced by Sharma-Taneja, Mittal, Borges-Roditi, and Kaniadakis-Lissia-Scarfone. The corresponding unique theorem is proved. It is shown that the obtained two-parameter Shannon additivity is a natural consequence from the Leibniz rule of the two-parameter Chakrabarti-Jagannathan difference operator.

I. INTRODUCTION

We often encounter complex systems which obey asymptotic power-law distribution in many fields such as high-energy physics, biophysics, turbulence, scale-free networks, economic science and so on. In order to explain the statistical natures of such systems, one of the fundamental approaches is a generalization of statistical mechanics in terms of a suitable generalization of the Boltzmann-Gibbs-Shannon (BGS) entropy. Tsallis' nonextensive thermostatistics [1, 2, 3, 4] is one of such generalizations. Naudts [5] has developed the generalized thermostatistics based on deformed exponential and logarithmic functions in general context.
In 1975 Sharma and Taneja [6], and independently Mittal [7] obtained a two parameter entropy in the field of information theory by generalizing Chaundy and McLeod's functional equation which characterizes Shannon' entropy. In the field of statistical physics, quite recently Kaniadakis, Lissia and Scarfone [8, 9] have considered a differential-functional equation imposed by the MaxEnt principle, and obtained the two-parameter (κ and r) entropy,

$$S_{\kappa,r} = -\sum_i p_i^{1+r}\left(\frac{p_i^{\kappa} - p_i^{-\kappa}}{2\kappa}\right), \tag{1}$$

which is equivalent to the Sharma-Taneja-Mittal entropy. For the sake of simplicity Boltzmann' constant k_B is set to unity in this paper. The two-parameter entropy $S_{\kappa,r}$ includes some one-parameter generalized entropies which proposed by Tsallis [10], by Abe [11] and by Kaniadakis [12] as a special case. For examples, when $r = \kappa$ and $q = 1 - 2\kappa$, $S_{\kappa,r}$ reduces to Tsallis' entropy

$$S_q = \frac{1 - \sum_i p_i^q}{q - 1}, \tag{2}$$

and when $r = 0$, $S_{\kappa,r}$ reduces to Kaniadakis' entropy

$$S_{\kappa} = \sum_i \frac{p_i^{1+\kappa} - p_i^{1-\kappa}}{2\kappa}. \tag{3}$$

Consequently the generalization of thermostatistics based on the two-parameter entropy provides us a unified framework of non-extensive thermostatistics. It has been shown that the two-parameter entropy has some important thermostatistical properties, such as positivity, continuity, expandability, concavity, Lesche stability, and so on [8, 9]. Thermodynamic stability for microcanonical systems described by the two-parameter entropy has been studied in Ref. [13]. Scarfone [14] has further developed the Legendre structure among the generalized thermal quantities in the thermostatistics based on the two-parameter entropy $S_{\kappa,r}$.

Abe [11] provided the procedure which generates an entropy functional from the function

$$g(s) \equiv \sum_i p_i^s, \tag{4}$$

where p_i is a probability of i-th event. He observed that the BGS entropy is obtained by acting the standard derivative on $g(s)$ as

$$\left[-\frac{dg(s)}{ds}\right]_{s=1} = -\sum_i p_i \ln p_i = S^{\mathrm{BGS}}, \tag{5}$$

whereas Tsallis' entropy is obtained by acting Jackson's q-derivative (or q-difference operator),

$$\mathrm{D}_x^q f(x) \equiv \frac{f(qx) - f(x)}{(q-1)x}, \tag{6}$$

as follows,

$$\left[-\mathrm{D}_s^q g(s)\right]_{s=1} = \frac{1 - \sum_i p_i^q}{q - 1} = S_q. \tag{7}$$

Johal [15] has established the connection between Tsallis entropy for a multifractal distribution and Jackson's q-derivative.

*Electronic address: wada@ee.ibaraki.ac.jp
†Electronic address: suyari@faculty.chiba-u.jp, suyari@ieee.org

Based on the same procedure as above, Borges and Roditi [16] has obtained the two-parameter generalized entropy,

$$\left[-\mathrm{D}_s^{\alpha,\beta} g(s)\right]_{s=1} = \sum_i \frac{p_i^\alpha - p_i^\beta}{\beta - \alpha} = S_{\alpha,\beta}, \tag{8}$$

by using the Chakrabarti and Jagannathan (CJ) difference operator [17]

$$\mathrm{D}_x^{\alpha,\beta} f(x) \equiv \frac{f(\alpha x) - f(\beta x)}{(\alpha - \beta)x}, \quad \alpha, \beta \in R. \tag{9}$$

The two-parameter CJ difference operator $\mathrm{D}_x^{\alpha,\beta}$ includes Jackson's q-derivative as a special case in which $\alpha = q$, and $\beta = 1$. The both two-parameter entropies Eqs. (1) and (8) are equivalent each other, and they are related by

$$\kappa = \frac{\beta - \alpha}{2}, \quad \text{and} \quad 1 + r = \frac{\alpha + \beta}{2}. \tag{10}$$

Note that Eq. (9) is symmetric under the interchange of the two parameters $\alpha \leftrightarrow \beta$. Consequently the two-parameter entropy $S_{\alpha,\beta}$ has the same symmetry.

On the other hand, it is well known that BGS entropy can be characterized by the Shannon-Khinchin (SK) axioms [18, 19]. During the rapid progress of Tsallis' thermostatistics, the generalized SK axioms were proposed by dos Santos [20] and by Abe [21]. Later, one of the authors [22] has generalized the SK axioms for one-parameter generalization of BGS entropy, and proved the uniqueness theorem rigorously. To the best of our knowledge, there is no generalization of SK axioms for either Kaniadakis' entropy S_κ or the two-parameter entropy $S_{\alpha,\beta}$. Since S_κ is a special case of $S_{\alpha,\beta}$, it is a natural to generalize the SK axioms for the two-parameter entropy $S_{\alpha,\beta}$. This is the main purpose of this paper. In the next section we review the one-parameter (q) generalization of SK axioms, among which the key ingredient is the q-generalized Shannon additivity. In order to develop a two-parameter generalization of the Shannon additivity, tree-graphical representation is utilized. In section III we prove the uniqueness theorem associated with the obtained two-parameter SK axioms. Some examples of a special case of the two-parameter entropy are presented in section IV. In section V it is shown that the two-parameter generalized Shannon additivity is symmetric under the interchange of the two parameters. The relation with the Leibniz rule of difference (or derivative) operator is discussed. Final section is devoted to our conclusion.

II. ONE-PARAMETER GENERALIZATION OF SHANNON ADDITIVITY

We first briefly review the q-generalized Shannon-Khinchin axioms [22], from which the following one-parameter (q) generalization of BGS entropy is uniquely determined:

$$S_q(p_1, \dots, p_n) = \frac{1 - \sum_{i=1}^n p_i^q}{\phi(q)}, \tag{11}$$

with $q \in R^+$ and $\phi(q)$ satisfies the following properties i)-iv):

i) $\phi(q)$ is continuous and has the same sign as $q - 1$;

ii) $\lim_{q \to 1} \phi(q) = 0$, and $\phi(q) \neq 0$ for $q \neq 0$;

iii) there exists an interval $(a, b) \in R^+$ such that $a < 1 < b$ and $\phi(q)$ is differentiable on the interval $(a, 1) \cup (1, b)$;

iv) there exists a positive constant k such that $\lim_{q \to 1} \frac{d\phi(q)}{dq} = \frac{1}{k}$.

The properties i)-iv) guarantee that Eq. (11) reduces to BGS entropy in the limit of $q \to 1$. In fact, by applying the l'Hopital's rule, we confirm that

$$\lim_{q \to 1} S_q = \lim_{q \to 1} \frac{-\sum_{i=1}^n p_i \ln p_i}{\frac{d\phi(q)}{dq}} = -k \sum_{i=1}^n p_i \ln p_i. \tag{12}$$

In physics k is Boltzmann' constant k_{B} (recall we set it unity in this paper), and in information theory k is a suitable constant to set the base of the logarithm, e.g, when $k = 1/\ln 2$, the base of the logarithm becomes two.

Let Δ_n be defined by the n-dimensional simplex

$$\Delta_n \equiv \left\{ (p_1, \dots, p_n) \middle| \; p_i \geq 0, \sum_{i=1}^n p_i = 1 \right\}. \tag{13}$$

The q-generalized SK axioms consist of the following four conditions:

- [GSK1] *continuity*: S_q is continuous in Δ_n and $q \in R^+$;
- [GSK2] *maximality*: for any $n \in N$ and any $(p_1, \dots, p_n) \in \Delta_n$

$$S_q(p_1, \dots, p_n) \leq S_q(\frac{1}{n}, \dots, \frac{1}{n}) \tag{14}$$

- [GSK3] *generalized Shannon additivity*: if

$$p_{ij} \geq 0, \quad p_i \equiv \sum_{j=1}^{m_i} p_{ij}, \quad p(j|i) \equiv \frac{p_{ij}}{p_i}, \\ \forall i = 1, \dots, n, \quad \forall j = 1, \dots, m_i \tag{15}$$

then the following equality holds

$$S_q(p_{11}, \dots, p_{nm_n}) = S_q(p_1, \dots, p_n) + \sum_{i=1}^n p_i^q S_q\left(p(1|i), \dots, p(m_i|i)\right) \tag{16}$$

- [GSK4] *expandability*:

$$S_q(p_1, \ldots, p_n, 0) = S_q(p_1, \ldots, p_n). \tag{17}$$

Note that when $q = 1$ the above axioms [GSK1]-[GSK4] reduce to the original SK axioms [19], respectively.

Shannon [18] discussed the synthesizing rule of an entropy with a tree-graphical representation. Let us now consider a further generalization of the axiom [GSK3] by utilizing the similar tree-graphical representation. Suppose we have a set of possible events (or choices), and let us divide each event (choice) into two successive sub-events (choices). Any joint probability of two successive sub-events can be expressed as

$$p_{ij} = p_i \; p(j|i), \tag{18}$$

where $p_i (i = 1, \ldots, n)$ is a probability of i-th sub-event and $p(j|i)$ a conditional probability, i.e., a probability of the j-th sub-event $(j = 1, \ldots, m_i)$ after the i-th sub-event occurred. More specifically, let us consider the following simple case in which $n = 2$ and $m_1 = 1, m_2 = 2$. Each probability of any event is graphically represented by a thin line as shown in Fig 1. Let S_q of Eq. (11) be ex-

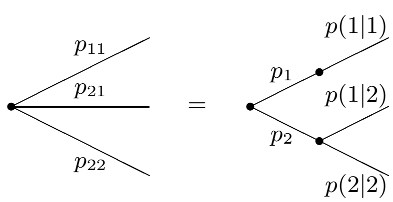

FIG. 1: A graphical representation of a set of two successive sub-events and associated probabilities Eq. (18) for $n = 2$ and $m_1 = 1, m_2 = 2$.

pressed as the following trace-form

$$S_q(p_1, \ldots, p_n) = \sum_{i=1}^{n} s_q(p_i), \tag{19}$$

where $s_q(p_i) = (p_i - p_i^q)/\phi(q)$. Then, for this simple case in Fig. 1, Eq. (16) in the axiom [GSK3] becomes

$$\sum_{i=1}^{2} \sum_{j=1}^{m_i} s_q(p_{ij}) = \sum_{i=1}^{2} \sum_{j=1}^{m_i} p_i^q \; s_q\left(p(j|i)\right) + \sum_{i=1}^{2} \sum_{j=1}^{m_i} s_q(p_i) \; p(j|i), \tag{20}$$

where $m_1 = 1$ and $m_2 = 2$. This can be graphically represented as Fig 2. A thick line represents the $s_q(r)$ of a probability r of the corresponding line, where r is p_{ij}, p_i or $p(j|i)$ depending on the line. A thin line represents a weight factor, which is either p_i^q for i-th sub-event or $p(j|i)$ for j-th sub-event). Summation over indices is represented by a node in each tree graph. Note that the

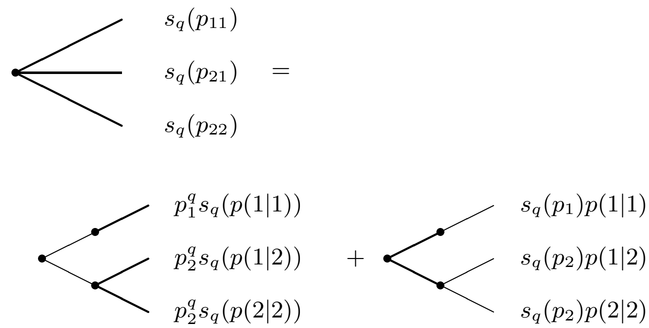

FIG. 2: A tree graphical representation for the specific example of the q-generalized Shannon additivity Eq. 20.

weight factor for a first sub-event of successive two sub-events is p_i^q whereas the weight factor for a second sub-event is $p(j|i)$. A natural extension of the one-parameter (q) generalized Shannon additivity in the axiom [GSK3] to the two-parameter entropy is then to attribute two different weighs to first and second sub-events, respectively. Hereafter we consider a generalized trace-form entropy

$$S_{\alpha,\beta}[p_i] = \sum_i s_{\alpha,\beta}(p_i), \tag{21}$$

depending on the two real-parameter α and β. Consequently a two-parameter generalization of Shannon additivity for the above simple example can be expressed as

$$\sum_{i=1}^{2} \sum_{j=1}^{m_i} s_q(p_{ij}) = \sum_{i=1}^{2} \sum_{j=1}^{m_i} p_i^\alpha \; s_q\left(p(j|i)\right) + \sum_{i=1}^{2} \sum_{j=1}^{m_i} s_q(p_i) \; p(j|i)^\beta. \tag{22}$$

Fig 3 is the graphical representation of Eq. (22).

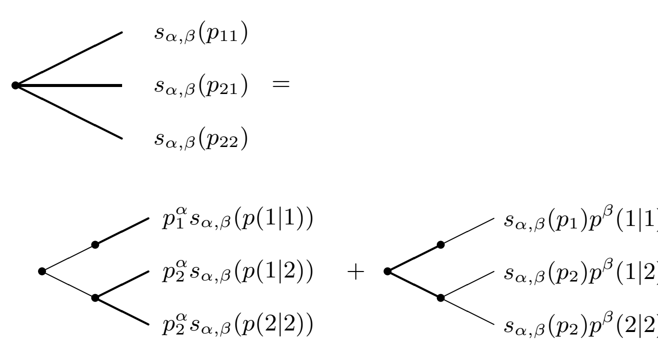

FIG. 3: A tree graphical representation of the two-parameter generalization of Shannon additivity for the simple case described in Fig. 1

III. TWO-PARAMETER GENERALIZATIONS OF SHANNON-KHINCHIN AXIOMS AND THE UNIQUENESS THEOREM

Now we propose the two-parameter generalization of the SK axioms, and prove the unique theorem.

Theorem: Let Δ_n be an n-dimensional simplex defined by Eq. (13). For a generalized trace-form entropy of Eq. (21), the following two-parameter generalized axioms [TGSK1]-[TGSK4] determine the function $S_{\alpha,\beta} : \Delta_n \to R^+$ such that

$$S_{\alpha,\beta}(p_1,\dots,p_n) = \sum_{i=1}^{n} \frac{p_i^\alpha - p_i^\beta}{C_{\alpha,\beta}}, \tag{23}$$

where α and β are real parameters restricted within the regions:

$$\begin{gathered}\Big\{(\alpha,\beta)\in R^2 |\ \alpha \geq 1, \beta \leq 1\Big\}\\ \Big\{(\alpha,\beta)\in R^2 |\ \alpha \leq 1, \beta \geq 1\Big\}\\ \text{but except } (\alpha,\beta)=(1,0) \text{ and } (0,1).\end{gathered} \tag{24}$$

and $C_{\alpha,\beta}$ satisfies the following properties I)-IV)

I) $C_{\alpha,\beta}$ is continuous w.r.t. α and β, and has the same sign as $\beta-\alpha$. Consequently $C_{\alpha,\beta}$ is anti-symmetric under the interchange of α and β, i.e., $C_{\beta,\alpha} = -C_{\alpha,\beta}$;

II) $\lim_{\alpha\to\beta} C_{\alpha,\beta} = 0$, and $C_{\alpha,\beta} \neq 0$ for $\alpha \neq \beta$;

III) there exists an interval $(a,b) \in R$ such that $C_{\alpha,\beta}$ is differentiable w.r.t. both α and β on the interval $(a,1)\cup(1,b)$;

IV) there exists a positive constant k such that $\lim_{\alpha\to 1}\frac{dC_{\alpha,\beta}}{d\alpha} = -\frac{1}{k}$, and $\lim_{\beta\to 1}\frac{dC_{\alpha,\beta}}{d\beta} = \frac{1}{k}$.

- [TGSK1]*continuity*: $S_{\alpha,\beta}$ is continuous in Δ_n;
- [TGSK2]*maximality*: for any $n \in N$ and any $(p_1,\dots,p_n)\in\Delta_n$

$$S_{\alpha,\beta}(p_1,\dots,p_n) \leq S_{\alpha,\beta}(\frac{1}{n},\dots,\frac{1}{n}) \tag{25}$$

- [TGSK3] *two-parameter generalized Shannon additivity*:

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{m_i} s_{\alpha,\beta}(p_{ij}) \ &= \ \sum_{i=1}^{n} p_i^\alpha \sum_{j=1}^{m_i} s_{\alpha,\beta}\left(p(j|i)\right)\\ &\quad + \sum_{i=1}^{n} s_{\alpha,\beta}(p_i)\sum_{j=1}^{m_i} p(j|i)^\beta.\end{aligned} \tag{26}$$

- [TGSK4] *expandability*:

$$S_{\alpha,\beta}(p_1,\dots,p_n,0) = S_{\alpha,\beta}(p_1,\dots,p_n). \tag{27}$$

Proof: First we consider the special case as same as Eq. (20) of Ref. [22], i.e., $\forall m_i = m$ and $p_{ij} = 1/(nm)$. Then Eq. (26) can be written as

$$\begin{aligned}nms_{\alpha,\beta}(\frac{1}{nm}) &= \sum_{i=1}^{n}\frac{1}{n^\alpha} ms_{\alpha,\beta}(\frac{1}{m}) + \sum_{i=1}^{n} s_{\alpha,\beta}(\frac{1}{n})\sum_{j=1}^{m}\frac{1}{m^\beta}\\ &= n^{1-\alpha} ms_{\alpha,\beta}(\frac{1}{m}) + nm^{1-\beta}s_{\alpha,\beta}(\frac{1}{n})\end{aligned} \tag{28}$$

Let $\lambda_{\alpha,\beta}(n)$ be defined by

$$\lambda_{\alpha,\beta}(n) \equiv -\frac{1}{n}\ s_{\alpha,\beta}(n), \tag{29}$$

then Eq. (28) becomes

$$\lambda_{\alpha,\beta}(\frac{1}{nm}) = n^{1-\alpha}\lambda_{\alpha,\beta}(\frac{1}{m}) + m^{1-\beta}\lambda_{\alpha,\beta}(\frac{1}{n}). \tag{30}$$

Exchanging the variables m and n, we have

$$\begin{aligned}n^{1-\alpha}\lambda_{\alpha,\beta}(\frac{1}{m}) &+ m^{1-\beta}\lambda_{\alpha,\beta}(\frac{1}{n})\\ &= m^{1-\alpha}\lambda_{\alpha,\beta}(\frac{1}{n}) + n^{1-\beta}\lambda_{\alpha,\beta}(\frac{1}{m}).\end{aligned} \tag{31}$$

The variable m and n are separated as

$$\frac{n^{1-\beta}-n^{1-\alpha}}{\lambda_{\alpha,\beta}(\frac{1}{n})} = \frac{m^{1-\beta}-m^{1-\alpha}}{\lambda_{\alpha,\beta}(\frac{1}{m})} = C_{\alpha,\beta}, \tag{32}$$

where $C_{\alpha,\beta}$ is a constant depending on α and β. We thus find

$$\lambda_{\alpha,\beta}(n) = \frac{n^{\beta-1}-n^{\alpha-1}}{C_{\alpha,\beta}}. \tag{33}$$

Next let us take p_{ij} as

$$p_{ij} = \frac{1}{\sum_{r=1}^{n} m_r}, \tag{34}$$

for all i and j, then

$$p_i = \sum_{j=1}^{m_i} p_{ij} = \frac{m_i}{\sum_{r=1}^{n} m_r}, \text{ and } p(j|i) = \frac{p_{ij}}{p_i} = \frac{1}{m_i}. \tag{35}$$

Eq. (26) becomes

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{m_i} s_{\alpha,\beta}\left(\frac{1}{\sum_{r=1}^{n} m_r}\right) &= \sum_{i=1}^{n} p_i^\alpha\sum_{j=1}^{m_i} s_{\alpha,\beta}\left(\frac{1}{m_i}\right)\\ &\quad + \sum_{i=1}^{n} s_{\alpha,\beta}(p_i)\sum_{j=1}^{m_i}\left(\frac{1}{m_i}\right)^\beta\end{aligned} \tag{36}$$

By utilizing Eqs (29) and (33) we have

$$
\begin{aligned}
\sum_{i=1}^{n} s_{\alpha,\beta}(p_i) m_i^{1-\beta} &= \sum_{i=1}^{n} p_i^{\alpha} \lambda_{\alpha,\beta}\left(\frac{1}{m_i}\right) - \lambda_{\alpha,\beta}\left(\frac{1}{\sum_{r=1}^{n} m_r}\right) \\
&= \frac{1}{C_{\alpha,\beta}}\left(\sum_{i=1}^{n} p_i^{\alpha} m_i^{1-\beta} - \left(\sum_{r=1}^{n} m_r\right)^{1-\beta}\right) \\
&\quad + \frac{1}{C_{\alpha,\beta}}\left(\left(\sum_{r=1}^{n} m_r\right)^{1-\alpha} - \sum_{i=1}^{n} p_i^{\alpha} m_i^{1-\alpha}\right)
\end{aligned}
\tag{37}
$$

From Eq. (35) it follows

$$
\sum_{i=1}^{n} p_i^t \, m_i^{1-t} = \left(\sum_{r=1}^{n} m_r\right)^{1-t}, \tag{38}
$$

with any real number t, then Eq. (37) becomes

$$
\sum_{i=1}^{n} s_{\alpha,\beta}(p_i) \, m_i^{1-\beta} = \sum_{i=1}^{n}\left(\frac{p_i^{\alpha} - p_i^{\beta}}{C_{\alpha,\beta}}\right) m_i^{1-\beta}. \tag{39}
$$

Since we can set m_i arbitrary, by setting $m_i = 1$ we finally obtain

$$
S_{\alpha,\beta}(p_1, \ldots, p_n) = \sum_i s_{\alpha,\beta}(p_i) = \sum_i \frac{p_i^{\alpha} - p_i^{\beta}}{C_{\alpha,\beta}}. \tag{40}
$$

Now we show that α and β are in the regions of Eq. (24) in order to the $S_{\alpha,\beta}$ is definite concave, i.e., the second derivative of $S_{\alpha,\beta}$ w.r.t. the p_i should be negative,

$$
\frac{d^2 S_{\alpha,\beta}}{dp_i^2} = \frac{\alpha(\alpha-1)p_i^{\alpha-2} - \beta(\beta-1)p_i^{\beta-2}}{C_{\alpha,\beta}} < 0. \tag{41}
$$

From the property I) we see $(\beta - \alpha)C_{\alpha,\beta}$ is always positive. Then the sign of the numerator multiplied by $\beta - \alpha$ should be negative, i.e.,

$$
(\beta - \alpha)\left\{\alpha(\alpha-1)p_i^{\alpha-2} - \beta(\beta-1)p_i^{\beta-2}\right\} < 0. \tag{42}
$$

Let us first consider a simple case in which one of the terms in the curly bracket of Eq. (42) is vanish. When $\beta = 1$, the condition becomes

$$
-\alpha(\alpha-1)^2 p_i^{\alpha-2} < 0. \tag{43}
$$

Since p_i is positive, then $\alpha > 0$.
When $\beta = 0$, the condition becomes

$$
-\alpha^2(\alpha-1)p_i^{\alpha-2} < 0, \tag{44}
$$

then $\alpha > 1$.
When $0 < \beta < 1$, the second term in the curly bracket is positive. Then the condition becomes $\beta - \alpha < 0$ and $\alpha(\alpha-1) > 0$. This is satisfied with $\alpha > 1$. The rest regions are obtained similar way because the condition of Eq. 42 is symmetric under the interchange of α and β.

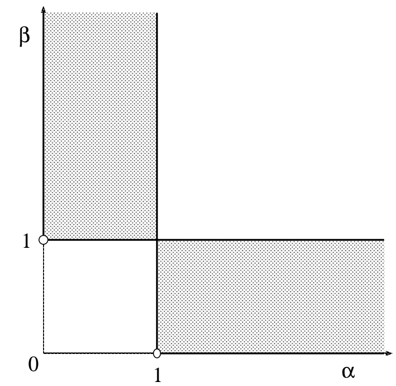

FIG. 4: The parameter regions in Eq. 24 of α and β in which the two-parameter entropy is definitely concave

The properties II)-IV) are needed in order to the $S_{\alpha,\beta}$ reduces to the BGS entropy in the limit of $\alpha, \beta \to 1$. In fact by applying l'Hopital's rule, we confirm that

$$
\begin{aligned}
\lim_{\alpha,\beta \to 1} S_{\alpha,\beta} &= \lim_{\alpha \to 1} \frac{-\sum_{i=1}^{n} p_i \ln p_i}{\frac{dC_{\alpha,\beta}}{d\beta}} \\
&= \lim_{\beta \to 1} \frac{\sum_{i=1}^{n} p_i \ln p_i}{\frac{dC_{\alpha,\beta}}{d\alpha}} = -k \sum_{i=1}^{n} p_i \ln p_i.
\end{aligned}
\tag{45}
$$

IV. SOME EXAMPLES OF A SPECIAL CASE

For the simplest case in which $C_{\alpha,\beta} = \beta - \alpha$, we see that

$$
\lambda_{\alpha,\beta}(x) = \frac{x^{\beta-1} - x^{\alpha-1}}{\beta - \alpha}, \qquad (x > 0). \tag{46}
$$

Recalling the relations (10) between the entropic parameters (α, β) and (κ, r), we see that $\lambda_{\alpha,\beta}(x)$ is nothing but the two-parameter deformed logarithmic function,

$$
\ln_{\{\kappa,r\}}(x) \equiv \frac{x^{r+\kappa} - x^{r-\kappa}}{2\kappa}, \tag{47}
$$

which is introduced in Ref. [8]. When $\alpha = 1 - \kappa$ and $\beta = 1 + \kappa$, the deformed logarithmic function reduces to κ-logarithmic function proposed by Kaniadakis.

$$
\lambda_{\alpha,\beta}(n) \to \frac{n^{\kappa} - n^{-\kappa}}{2\kappa} = \ln_{\{\kappa\}} n. \tag{48}
$$

The entropy $S_{\alpha,\beta}$ reduces to Kaniadakis' entropy Eq. (3). When $\alpha = q$ and $\beta = 1$, it reduces to Tsallis' q-logarithmic function but q replaced with $2 - q$

$$
\lambda_{\alpha,\beta}(n) \to \frac{n^{q-1} - 1}{q - 1} = \ln_{2-q}(n). \tag{49}
$$

Accordingly $S_{\alpha,\beta}$ reduces to Tsallis' entropy Eq. (2). More details on the two-parameter deformed logarithms and entropies, please refer to Ref. [8].

Another example is Harvda-Charvat [23] or Daróczi [24] entropy,

$$S_q^{\mathrm{HCD}} = \frac{1-\sum_i p_i^q}{1-2^{1-q}}, \tag{50}$$

which corresponds to the case $C_{\alpha,\beta} = 1-2^{1-\alpha}$, $\alpha = q, \beta = 1$ and $k = 1/\ln 2$.

V. ON THE TWO-PARAMETER GENERALIZED SHANNON ADDITIVITY

Since $C_{\beta,\alpha} = -C_{\alpha,\beta}$, it is obvious from Eq. (23) that $S_{\alpha,\beta}$ (and $s_{\alpha,\beta}$) is symmetric under the interchange of the two-parameter α and β. Then the two-parameter generalized Shannon additivity Eq. (26) also must hold if α and β are interchanged each other,

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{m_i} s_{\alpha,\beta}(p_{ij}) \;=\; & \sum_{i=1}^{n} p_i^{\beta} \sum_{j=1}^{m_i} s_{\alpha,\beta}\left(p(j|i)\right) \\ & + \sum_{i=1}^{n} s_{\alpha,\beta}(p_i) \sum_{j=1}^{m_i} p(j|i)^{\alpha}. \end{aligned} \tag{51}$$

Then by adding the both sides of Eqs. (26) and that of (51) (and dividing by 2) we obtain the symmetric form

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{m_i} s_{\alpha,\beta}(p_{ij}) = & \sum_{i=1}^{n} \iota_{\alpha,\beta}(p_i) \sum_{j=1}^{m_i} s_{\alpha,\beta}\left(p(j|i)\right) \\ & + \sum_{i=1}^{n} s_{\alpha,\beta}(p_i) \sum_{j=1}^{m_i} \iota_{\alpha,\beta}(p(j|i)), \end{aligned} \tag{52}$$

where we introduced the function

$$\iota_{\alpha,\beta}(x) \equiv \frac{x^{\alpha}+x^{\beta}}{2}. \tag{53}$$

Recall Eq. (8) in which the two-parameter entropy $S_{\alpha,\beta}$ (and $s_{\alpha,\beta}(p_i)$) is obtained by acting the CJ difference operator on $g(s)$ of Eq. (4). Similarly the function $\iota_{\alpha,\beta}(p_i)$ is obtained from $g(s)$ by acting the average operator $M_x^{\alpha,\beta}$ associated with CJ difference operator

$$M_x^{\alpha,\beta} f(x) \equiv \frac{f(\alpha x)+f(\beta x)}{2}, \tag{54}$$

as follows,

$$\left[M_s^{\alpha,\beta} g(s)\right]_{s=1} = \sum_i \frac{p_i^{\alpha}+p_i^{\beta}}{2} = \sum_i \iota_{\alpha,\beta}(p_i). \tag{55}$$

From the relations in Eq. (10) we see that this is same as the function

$$\mathcal{I}_{\kappa,r} \equiv \sum_i p_i^{r+1}\left(\frac{p_i^{\kappa}+p_i^{-\kappa}}{2}\right), \tag{56}$$

which is introduced in Ref. [14], and an important quantity relating the two-parameter generalized entropy of Eq. (1), free-energy, partition function, and other thermodynamical quantities.

With the help of the average operator $M_x^{\alpha,\beta}$, the Leibniz rule of the CJ difference operator can be written in the symmetric form as

$$\begin{aligned}\mathrm{D}_x^{\alpha,\beta}\Big(f(x)g(x)\Big) = & \Big(\mathrm{D}_x^{\alpha,\beta} f(x)\Big)\Big(M_x^{\alpha,\beta} g(x)\Big) \\ & + \Big(M_x^{\alpha,\beta} f(x)\Big)\Big(\mathrm{D}_x^{\alpha,\beta} g(x)\Big). \end{aligned} \tag{57}$$

Then we observe that the two-parameter generalized Shannon additivity (52) is readily obtained by acting $\mathrm{D}_s^{\alpha,\beta}$ on $\sum_i\sum_j p_{ij}^s = \sum_i p_i^s \sum_j p(j|i)^s$, as can be seen from the relation

$$\begin{aligned}\sum_{i,j}\left[\mathrm{D}_s^{\alpha,\beta} p_{ij}^s\right]_{s=1} = & \sum_i \left[\mathrm{D}_s^{\alpha,\beta} p_i^s\right]_{s=1} \sum_j \left[M_s^{\alpha,\beta} p(j|i)^s\right]_{s=1} \\ & + \sum_i \left[M_s^{\alpha,\beta} p_i^s\right]_{s=1} \sum_j \left[\mathrm{D}_s^{\alpha,\beta} p(j|i)^s\right]_{s=1}. \end{aligned} \tag{58}$$

Thus we see that the two-parameter Shannon additivity is a natural consequence of the Leibniz rule of the CJ difference operator. In other words, the Shannon additivity associated with an entropy is the consequence of the Leibniz rule of the corresponding difference (or derivative) operator which generates the entropy.

Finally let us comment on the number of the parameters for generalizing the BGS entropy. One may wonder whether a generalization to more than two parameters is possible or not. We can answer to this question as follows. Recall that a parameter generalization of the BGS entropy is obtained by acting a *first-order* difference operator on the function $g(s)$, e.g., Eq. (7) for the one-paramere entropy S_q and Eq. (8) for the two-parameter entropy $S_{\alpha,\beta}$. Since any first-order difference operator is defined by the difference of the functions at two points (e.g., Eq. (9) for $\mathrm{D}_x^{\alpha,\beta}$), such a generalization of the BGS entropy is up to two parameters.

VI. CONCLUSION

Based on the one-parameter generalized SK axioms [22] proposed by one of the authors, we have further developed the two-parameter generalization of the SK axioms in accordance with the two-parameter entropy introduced by Sharma-Taneja [6], Mittal [7], Borges-Roditi [16], and Kaniadakis-Lissia-Scarfone [9], and proved the corresponding uniqueness theorem. The Shannon additivity, which is a key ingredient of the SK axioms, is generalized by considering the tree-graphical representation. We have obtained the symmetric form of the two-parameter generalized Shannon additivity, and shown the relation with the Leibniz rule of the CJ difference operator.

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